# Self-focusing arrest of femtosecond laser pulses in air at different pressures

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We study analytically and numerically the self-focusing arrest of femtosecond laser pulses in air at different pressures in the presence of an external focus lens. Analytical estimations as well as results of simulations show that the intensity at which the self-focusing arrest occurs is almost independent of the gas pressure. However, a dependence on the temperature is found. The Raman effect is taken into account, and an estimation of the intensity inside of filaments at high altitudes is given.

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## INTRODUCTION

The nonlinear propagation of femtosecond laser pulses through air produces self-guided high-intensity plasma filaments for laser input powers above some critical power [1]. The filamentation process starts at the self-focusing distance, i.e., the point where the initial self-focusing process of the pulse is arrested by the defocusing effect of the plasma generated via multiphoton ionization of the air. Beyond the self-focusing distance the laser pulse undergoes strong temporal and spatial changes which are reflected in the nonlinear dynamics of the filamentation process. We note that the x-wave generation has been observed during the filamentation process in transparent condensed media [2], where normal group-velocity dispersion plays a more dominant role in comparison to air. In view of the perspectives of atmospheric applications such as remote sensing [3] and lightning control [4], it is necessary to study filamentation in air under atmospheric conditions, e.g., at different pressures. The dependence on the gas pressure has been investigated in a few theoretical [5,6] and experimental [7,8] studies up to now.

In this paper we present a study of the value of the intensity of the laser pulse at the self-focusing distance. We show that this value is rather independent of the pressure and input power of the laser pulse. We provide a closed-form expression of this intensity, which takes into account the effect of the Raman nonlinearity. This expression agrees well with simulations and shows that the temperature is an important parameter to estimate this intensity. Moreover, since the intensity at the self-focusing point is of the same order of magnitude as that observed inside of filaments [9], we make some estimates of the intensity of filaments at high altitudes in the atmosphere where direct measurements are difficult to perform.

We note that the filamentation in air is a rather complex phenomenon and no analytical solution exists. However, the initial self-focusing process in gases is still analytically tractable [10–12]. Using a variational theory we recently derived [10,13] semianalytical estimations of the self-focusing distance, the maximum intensity and the radius of the high-intensity core of the pulse during the self-focusing process. These estimations are shown to provide a better agreement with results of numerical simulations than previous analytical theories. The improvement is due to the effect of the low-intensity background, propagating with the high-intensity inner core during the self-focusing process [14]. This background has been included as a first-order correction in the phase of the trial solution in our ansatz. Below, we use this analytical theory to derive a closed-form expression of the intensity of the laser pulse at the self-focus for different pressures.

## I. MODEL

A theoretical analysis of the propagation of an intense femtosecond laser pulse in air at high pressure (>1 Torr) requires the inclusion of the spatiotemporal dynamics of the pulse. This is done by a set of coupled equations, which are written in a dimensionless form in the retarded coordinate system $(t \to t - z/v_g)$ as

$$i\partial_\xi \mathcal{E} + \frac{1}{4}\Delta_\perp \mathcal{E} + \frac{L_D}{\tilde{L}_{NL}}\delta n \mathcal{E} - \frac{1}{4}\frac{L_D}{L_d}\left(\partial_\tau^2 \mathcal{E} + \frac{i}{3}\frac{L_d}{L_d'}\partial_\tau^3 \mathcal{E}\right) - \frac{L_D}{L_{plas}}\rho\mathcal{E} + i\frac{L_D}{L_{AP}}\rho\mathcal{E} + i\frac{L_D}{L_{MPA}}|\mathcal{E}|^{2n-2}\mathcal{E} = 0, \quad (1)$$

where $\mathcal{E}$ is the electric field normalized to its initial maximum $\sqrt{I_0}$ and $I_0$ is the input peak intensity given in units of $W/cm^2$. $\rho$ is the density of electrons normalized to the initial number density of molecules $N_0$, and obeys the relation

$$\rho = 1 - \exp\left(-T_0\sigma^{(n)}I_0^n\int_{-\infty}^{\tau}|\mathcal{E}|^{2n}d\tau\right). \quad (2)$$

In Eq. (1)

$$\delta n = \delta n_{Kerr} + \delta n_{Raman}, \quad (3)$$

where $\delta n_{Kerr} = (1-\alpha)|\mathcal{E}|^2$ is the instantaneous Kerr nonlinearity and

$$\delta n_{Raman} = \alpha\int_{-\infty}^{\tau} R(\tau - \tau')|\mathcal{E}|^2 d\tau' \quad (4)$$

is the Raman nonlinearity [15] which counts for the delayed Kerr response of air. Experimental results of propagation fs-laser pulses in air with $\lambda = 800$ nm suggest that the fraction of the nonlinear optical response $\alpha = 0.5$ [15]. In fact, in Ref. [15] it is shown the exact behavior of $R(\tau)$, which initially

 

undergoes a damped harmonic motion followed by spontaneous resurgences at times longer than 2 ps at room temperature (300 K). For lower temperatures, namely 50 and 100 K, we have observed, by using the formulation in Ref. [15], that the resurgences appear at times of about 1.5 ps. So these resurgences are not relevant for a single pulse of few hundred fs or shorter. In this case, the response function can be modeled [16] by

$$R(\tau) = \frac{\gamma^2 + 4\Lambda^2}{4\Lambda} \exp\left(-\frac{\gamma\tau}{2}\right)\sin(\Lambda\tau), \tag{5}$$

where $\Lambda$ and $\gamma$ are the characteristic Raman frequency and phenomenological damping rate, respectively. The values of $\Lambda$ and $\gamma$, which depend on the temperature, can be obtained by fitting the exact response function given in [15].

Equation (1) has radial symmetry, in which the radius $r$ and the time $\tau$ are given in units of length $w_0$ and duration $T_0$ of the laser pulse, respectively, the longitudinal coordinate $\xi$ is given in units of the diffraction length scale $L_D = k_z w_0^2/2$, where $k_z = n_0 k_0$. Here, $n_0 = 1. + 0.000264 p/p_{atm}$ is the linear refraction index [17], $p$ is the pressure of the gas medium, and $p_{atm} = 760$ Torr is the value of the atmospheric pressure at the sea level. $k_0 = 2\pi/\lambda$, where the wavelength $\lambda = 800$ nm. The dispersion length scales are defined as $L_d = T_0^2/(2k_2)$ and $L_{d'} = T_0^3/(2k_3)$, where $k_2$ and $k_3$ are the second and third order group-velocity dispersion (GVD) coefficients, respectively. The value of these coefficients at 800 nm is given by the expressions $k_2 = (0.20\ \text{fs}^2/\text{cm}) p/p_{atm}$, $k_3 = (0.35\ \text{fs}^3/\text{cm}) p/p_{atm}$ [17]. In Eq. (1) the coefficient $L_D/\tilde{L}_{NL} = 2P/\tilde{P}_{cr}$, where $P$ is the input power and $\tilde{P}_{cr}$ is the critical power in the continuous-wave limit, which is about 3 GW for air at atmospheric pressure. The plasma length scale $L_{plas} = k_z m_e c^2/(2\pi e^2 N_0)$, the absorption-plasma length scale $L_{AP} = m_e \omega c^2 \tau_c k_z/(2\pi e^2 N_0)$, and the multiphoton-absorption length scale $L_{MPA} = 2/(n\hbar\omega\sigma^{(n)} I_0^{n-1} N_0)$. $m_e$ is the electron mass, $\omega = ck_0$, where $c$ is the speed of the light. The multiphoton ionization transition rates are obtained by fitting the experimental data to the form $\sigma^n |\mathcal{E}|^2$, which is valid in the intensity range up to $2 \times 10^{14}$ W/cm$^2$. It yields $\sigma^{(n)} = 7.46 \times 10^{-72} (\text{cm}^2/\text{W})^n$ with $n = 5.93$.

## II. SEMIANALYTICAL ESTIMATION: INTENSITY AT THE SELF-FOCUS

In order to derive a semianalytical expression for the intensity at the self-focusing distance we first approximate Eq. (2) by [13,19]

$$\rho = T_0 \sigma^{(n)} I_0^n \int_{-\infty}^{\tau} |\mathcal{E}|^{2n} d\tau \ll 1. \tag{6}$$

This approximation is valid since the ionization probability in laser filaments is well below unity (e.g., [18]). Equation (6) can be integrated by using a simple integration rule up to the peak of the pulse [13,19], as $\rho = g(\tau)|\mathcal{E}|^{2n}$, where $g(\tau) = (\tau_{min} + \tau) T_0 \sigma^{(n)} I_0^n/2$. Here, $\tau_{min}$ is a cutoff determined by the initial pulse.

In order to simplify the term $\delta n_{Raman}$ given in Eq. (4) we assume that during the self-focusing process the temporal part of the intensity, $|\mathcal{E}|^2$, remains Gaussian with a duration $T_0$. We note that this approach is valid, since the self-focusing is mostly a spatial process [10,11]. In this case, $\delta n_{Raman}$ can be calculated analytically, yielding a function of time, duration of the pulse, and temperature. In the case of femtosecond pulses, we observe a delay of the amplitude $\delta n_{Raman}$ with respect to the amplitude of the instantaneous Kerr nonlinearity $\delta n_{Kerr}$ (see Ref. [12], and figures therein). In the case of pulse durations $T_0 > 1ps$, this delay tends to be zero and the form of $\delta n_{Raman}$ tends to be equal to the form of $\delta n_{Kerr}$. We note that most of the physics in the self-focusing process is governed by the maximum intensity of the pulse [10,13,19], which is located at $\tau = 0$ here. So, a further simplification of $\delta n_{Raman}$ can be done byassuming that only its amplitude around the center of the pulse ($\tau = 0$) participates effectively in the self-focusing process, so delayed terms are neglected. For convenience, we define an effective Raman nonlinearity of the form $\delta n_{Raman} = \alpha R_0 |\mathcal{E}|^2$, where the term

$$R_0 = i \exp\left[\tfrac{1}{32} T_0^2 (\gamma - 2i\Lambda)^2\right] \frac{\sqrt{\pi/2}\ T_0(\gamma^2 + 4\Lambda^2)}{16\Lambda} \times \left[ \operatorname{erf}\left(\frac{T_0(\gamma - 2i\Lambda)}{4\sqrt{2}}\right) + \exp\left(\tfrac{1}{4} i T_0^2 \gamma\Lambda\right) - \exp\left(\tfrac{1}{4} i T_0^2 \gamma\Lambda\right) \operatorname{erf}\left(\frac{T_0(\gamma + 2i\Lambda)}{4\sqrt{2}}\right) + 1 \right], \tag{7}$$

is the leading term of the Taylor series expansion of $\delta n_{Raman}/(\alpha|\mathcal{E}|^2)$ around $\tau = 0$. $R_0$ in Eq. (7) is real and the function $\operatorname{erf}(z) = 2/\sqrt{\pi}\int_0^z \exp(-t^2) dt$. Finally, $\delta n$ given in Eq. (3) takes the form: $\delta n = (1 - (1 - R_0)\alpha)|\mathcal{E}|^2$, where the real coefficient $1 - (1 - R_0)\alpha$ is plotted in Fig. 1 for different values of the temperature and duration of the pulse. Note that for durations $T_0 < 200$ fs this coefficient $1 - (1 - R_0)\alpha \to 0.5$, i.e., the pulse is so short that only the instantaneous Kerr effect counts for the pulse propagation. For long pulses ($T_0 > 1$ ps) the coefficient $1 - (1 - R_0)\alpha \to 1$, i.e., both instantaneous Kerr nonlinearity and Raman nonlinearity are equally important and cannot be distinguished during propagation. Note that for a given duration of the pulse the effect of the Raman nonlinearity is stronger for low temperatures (Fig. 1: solid line) than for room temperature (Fig. 1: dotted line).

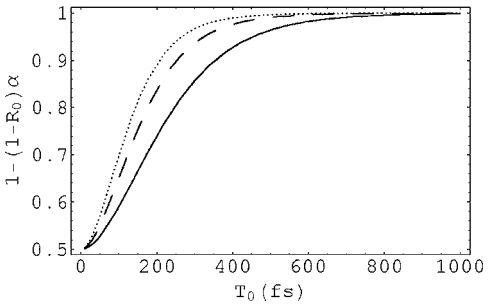


FIG. 1. $1 - (1 - R_0)\alpha$ vs. duration of the pulse $T_0$ for three different temperatures, namely 100 K (solid line), 200 K (dashed line), and 300 K (dotted line).

We have observed in our simulations that during the self-focusing process the GVD effect is negligible in the presence of an external focus lens $f$ for $f<z_{sf}$. Here, $z_{sf}$ is the self-focusing distance in the absence of the external focus lens ($f=\infty$). In fact, in the case of $f=\infty$ the GVD effect causes an increment of the self-focusing distance and a reduction of the intensity at the self-focusing point. In the present paper we consider the presence of an external focus lens, so we further neglect the GVD effect in our analytical estimations. So Eq. (1) can be reduced to

$$i\partial_{\xi}\mathcal{E}+\frac{1}{4}\Delta_{\perp}\mathcal{E}+\frac{L_D}{L_{NL}}|\mathcal{E}|^2\mathcal{E}-\frac{L_D}{L_{plas}}g(\tau)|\mathcal{E}|^{2n}\mathcal{E}$$
$$=-i\left(\frac{L_D}{L_{AP}}g(\tau)|\mathcal{E}|^{2n}+\frac{L_D}{L_{MPA}}|\mathcal{E}|^{2n-2}\right)\mathcal{E}, \tag{8}$$

where $L_D/L_{NL}=2P/P_{cr}$ and

$$P_{cr}=\frac{\tilde{P}_{cr}}{1-(1-R_0)\alpha} \tag{9}$$

is the critical power expression in the presence of the Raman nonlinearity. Note in Eq. (9) that in the absence of Raman nonlinearity ($\alpha=0$) $P_{cr}=\tilde{P}_{cr}$. On the other hand, by definition $\tilde{P}_{cr}=\lambda^2/(2\pi n_o n_2)$, where $n_2$ is the nonlinear refractive-index coefficient due to the bound electrons of the molecules. Note that $n_2$ depends linearly on the density of the gas, and therefore on the pressure, $p$. So, $\tilde{P}_{cr}$ depends linearly on the inverse pressure $p^{-1}$ [5], and can be written in the form $\tilde{P}_{cr}=\tilde{P}_{cr}^{(atm)}p_{atm}/p$, where $\tilde{P}_{cr}^{(atm)}=3$ GW is the value of the critical power of the air at atmospheric pressure, i.e., when $p=p_{atm}$

In order to obtain approximated analytical solutions of Eq. (8) we apply a variational method, which has been already outlined in several papers [10,13,19] for the case of an initial Gaussian pulse, namely

$$\mathcal{E}(\xi=0,r,\tau)=\exp(-\tau^2)\exp(-r^2)\exp(ib_0r^2). \tag{10}$$

Here, $b_0$ is a constant describing the initial wave front divergence of the laser pulse, i.e., $b_0=0$ for a collimated pulse, and $b_0=-1/f$ in the case of an external lens with focal length $f$ in units of $L_D$. Eventually we can get a set of ordinary differential equations in $\xi$, whose analytical solution provides a closed-form expression of the intensity of the pulse at the self-focus, namely,

$$I_{arrest}=\frac{1}{n_0}\left(\frac{[1-(1-R_0)\alpha]c^2k_Bm_e\Theta(1+n)^2}{2e^2\sigma^{(n)}T_0\tilde{P}_{cr}^{(atm)}p_{atm}}\right)^{1/(n-1)}. \tag{11}$$

Here, $k_B$ is the Boltzmann's constant. It is interesting to note that $I_{arrest}$, Eq. (11), is a function of the temperature, $\Theta$, given in kelvins, so *not* explicitly dependence on the pressure is observed. Of course, one can alternatively use the ideal gas law, $N_0 k_B\Theta=p$, to write $I_{arrest}$ in terms of the pressure $p$ and the initial number of molecules $N_0$. However, in this alternative case one has to calculate, in addition, $N_0$ for every value of the pressure by using again the ideal gas law.

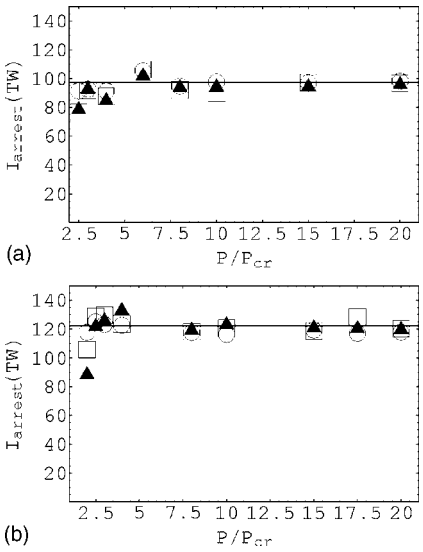


FIG. 2. Intensity at the self-focus as a function of input power $P$ (scaled in units of the critical power $P_{cr}$). Results of numerical simulations for a temperature of 300 K at different gas pressures, 100 Torr (open circles), 400 Torr (open squares), and 760 Torr (filled triangles), are compared with an analytical estimation $I_{arrest}$ (solid line) given by Eq. (11). (a), $T_{FWHM}=250$ fs and $I_{arrest}=97.28$ TW/cm$^2$; (b), $T_{FWHM}=50$ fs and $I_{arrest}=122.02$ TW/cm$^2$.

So, in principle it is straightforward to write $I_{arrest}$ in Eq. (11) in terms of the temperature $\Theta$ and the duration $T_0$ of the pulse.

## III. RESULTS AND DISCUSSION

We have solved numerically Eqs. (1)–(5) with the initial condition (10) using the Crank-Nicholson scheme [20]. For the results in Fig. 2 we have considered propagation of a 800 nm laser pulse in air at three values of pressure at room temperature ($\Theta=300$ K), namely $p=100$, 400, and 760 Torr. We consider the presence of an external focus lens with $f=100$ cm. The temporal and spatial form of $\mathcal{E}(r,\tau)$ have been chosen to be Gaussian with length $w_0=0.39$ cm and duration $T_0=212$ fs ($T_{FWHM}=\sqrt{2\ln 2}T_0=250$ fs). The complex pulse has been represented on a regular grid. The typical grid sizes were $6w_0$ in the radial direction and $10T_0$ in the time domain and the propagation step size has been chosen adaptive to ensure that the relative error of the on-axis intensity peak of the pulse did not exceed $10^{-6}$.

The analytical estimations are found to be in good agreement with results of numerical simulations. As an example we compare in Fig. 2 the value of $I_{arrest}$, given by Eq. (11), and the results following from numerical simulations at different pressures of the air and durations of the laser pulse. We observe that for a given duration of the pulse the intensity at the self-focusing point, following from simulations, is rather independent of the pressure for high input powers ($P>2P_{cr}$), as predicted in Eq. (11).

On the other hand, we note that despite the large variation (five times) of the duration of the pulse between Figs. 2(a) and 2(b), the relative variation of the intensity

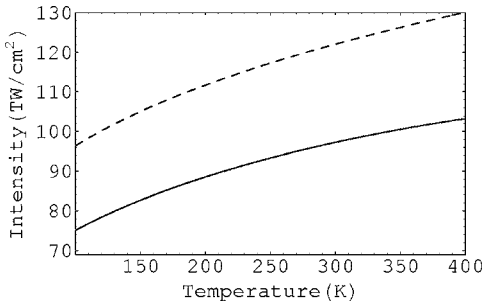


FIG. 3. $I_{arrest}$ vs temperature for $T_{FWHM}=50$ fs (dashed line) and $T_{FWHM}=250$ fs (solid line).

$(\Delta I_{arrest}/I_{arrest})$ is small, about 20%. According to Eq. (11), this small relative variation is due to the polynomial dependence of the intensity on the duration of the pulse, i.e., $I_{arrest} \sim T_0^{-1/(n-1)} = T_0^{-0.2}$. Note that we can expect a similar dependence on the duration for other forms of single pulses. In fact, during filamentation, when the pulse undergoes temporal splitting to two pulses [21], the duration of the new single pulses are shorter than the original one, therefore, higher intensities beyond the self-focusing distance can be observed [13]. However, the relative variation of the maximum intensity beyond the self-focusing distance, during filamentation, is small, about 20%, as can be observed in simulations (see, e.g., Ref. [13], and figures therein).

We note that the self-focusing point in our numerical results has been defined as the first local maximum of the on-axis electron density. The nonsmooth behavior of the intensity evaluated from numerical results is, in principle, due to the highly nonlinear behavior of the electron density. Here uncertainties arising from measurements are negligible since they are proportional to the sampling step size, which is small.

We note also that the value of $I_{arrest}$ found here is about twice larger than the empirical prediction of the arrest intensity given in Ref. [9].

In view of the potential application of filamentation in the atmosphere, it is most interesting to study $I_{arrest}$ depending on the temperature. In Fig. 3 is shown the estimation of $I_{arrest}$ as a function of the temperature $\Theta$ for two pulse durations. We observe that $I_{arrest}$ increases monotonically with the temperature, however, its relative variation is small, about 17% between 100 and 300 K. In fact, if we consider real atmospheric conditions, where the temperature of the air depends on the altitude, we observe, for instance, that the relative variation of the intensity between the sea level ($\Theta \simeq 300$ K) and the upper part of the troposphere, at about 17 km of altitude ($\Theta \simeq 130$ K), is about 10%. This is in a qualitative agreement with recent experimental observations [8], which show that the filamentation process is qualitatively not affected at high altitudes under real atmospheric conditions.

## IV. CONCLUSION

In conclusion, we have derived an analytical expression for the intensity of femtosecond laser pulses at the self-focusing point. The Raman nonlinearity has been taken into account. The estimations of the theory are in a quantitative agreement with the results of numerical simulations over a broad range of input powers. We observe both numerically and analytically that the intensity of the pulse at the self-focusing distance does not depend explicitly on the pressure of the air. In view of potential applications in the atmosphere it is most interesting that the theory predicts that the intensity depends on the temperature. Ranges for the intensity at the self-focusing point at different temperatures are given. In general, it is found that under atmospheric conditions variations in maximum intensities do not vary significantly with the altitude. This analysis is in agreement with recent experimental observations in filamentation.

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